Stable equilibrium distribution of a LV system: An Approximate Message Passing approach

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Model and general problem

The model

Abundances of n living species $u : \mathbb{R}_+ \mapsto \mathbb{R}_+^n$ follow the generalized Lotka-Volterra ODE

$$\dot{u}(t) = u(t) \odot (r + (Y - I)\, u(t))$$

- $Y \in \mathbb{R}^{n \times n}$: **interaction matrix**
- $r \in \mathbb{R}_+^n$: **intrinsic growth rates** vector

Problem

The LV ODE has a **globally stable equilibrium** $u_\star = \left[u_{\star,i} \right] \in \mathbb{R}_+^n$
if $\exists$ a diagonal matrix $C > 0$ such that $CY + Y^\top C < 2C^2$.
We assume that

- Y and r are **random**, $r \perp\!\!\!\perp Y$
- $n \to \infty$
- $\limsup_n \|(Y + Y^\mathsf{T})/2\| < 1$ a.s.

Problem

Asymptotics of the **equilibrium distribution**, *i.e.*, the random probability measure

$$\mu^{u_\star} = \frac{1}{n} \sum_{i \in [n]} \delta_{u_{\star,i}}$$

Usually $u_\star \in$ border of 1st quadrant. Percentage of surviving species ?

LCP characterization of equilibrium

When $\|(Y + Y^\top)/2\| < 1$, matrix $I - Y$ is a P–matrix, *i.e.*, all principal minors > 0. Then

- The **Linear Complementarity Problem** (LCP): here, find a vector z such that

$$
\begin{aligned}
z &\succcurlyeq 0 \\
z \odot (r + (Y - I)\, z) &= 0 \\
r + (Y - I)\, z &\preccurlyeq 0
\end{aligned}
$$

  has an unique solution for each $r \in \mathbb{R}^n$
- $u_\star$ **is this solution**

Our approach

Asymptotics of $\mu^{u_\star}$ when $u_\star = \mathrm{LCP}(Y - I, r)$

Interaction matrix from Gaussian Orthogonal Ensemble (GOE)

LV in GOE case

LV model

$$\dot{u} = u \odot (r + (\beta G - I)\, u)$$

where

- G is GOE.
- $r \perp\!\!\!\perp G$, empirical measure $\mu^r \xrightarrow{\text{a.s.}} \bar{\mu}_r$ is the Wasserstein space $\mathcal{P}_2(\mathbb{R}_+)$
- We take $\beta < 1/2$. Indeed, $\|G\| \xrightarrow{\text{a.s.}}_n 2$ as a GOE matrix. Thus $u_\star$ exists and the LCP problem is well-defined since $\limsup_n \beta \|G\| < 1$ a.s.

$\mu^{u_\star}$ asymptotics in GOE case

Theorem [AHMN'23]

Let $R \sim \bar{\mu}_r$ and $Z \sim \mathcal{N}(0,1)$ with $R \perp\!\!\!\perp Z$. For each $\beta < 1/\sqrt{2}$, the system

$$\beta = \frac{\alpha}{1+\gamma\alpha^2}$$
$$\sigma^2 = \alpha^2 \mathbb{E}(\sigma Z + R)_+^2$$
$$\gamma = \mathbb{P}\left[\sigma Z + R > 0\right]$$

admits an unique solution $(\alpha, \sigma, \gamma) \in (\sqrt{2}, \infty) \times (0, \infty) \times (0, 1)$. Moreover, for $\beta < 1/2$ ($< 1/\sqrt{2}$ *in physics literature*)

$$\mu^{u_\star} \xrightarrow[n\to\infty]{\text{a.s.}} \mathscr{L}\left((1+\alpha^2\gamma)(\sigma Z + R)_+\right) \quad \text{in } \mathcal{P}_2(\mathbb{R}_+)$$

Note: γ is (a lower bound on) the limit proportion of surviving species

Result obtained previously by [Bunin'17], [Galla'18], ...

Proof by Approximate Message Passing (AMP)

AMP principle

Iterative algorithm widely studied in statistical physics, coding and wireless communications, learning theory, ...

Basic algorithm: G is a $n \times n$ GOE matrix, and $f_0, f_1, \ldots$ is a sequence of $\mathbb{R}^2 \to \mathbb{R}$ Lipschitz functions

General algorithm form

$$x_{k+1} = \Big[x_{k+1,i}\Big]_{i\in[n]} = Gf_k(x_k, a) + \text{"correction"}, \quad f_k(x_k, a) = \Big[f_k(x_{k,i}, a_i)\Big]_i$$

where a is a parameter vector, and $(x_0, a) \perp\!\!\!\perp G$

Thanks to the correction, we can identify the asymptotics of the **joint empirical measure**

$$\mu^{a,x_1,x_2,\ldots,x_k} = \frac{1}{n} \sum_{i\in[n]} \delta_{a_i,x_{1,i},x_{2,i},\ldots,x_{k,i}} \in \mathcal{P}(\mathbb{R}^{k+1})$$

when $n \to \infty$, for each fixed k

AMP algorithm and results

AMP algorithm:

$$x_{k+1} = Gf(x_k, a) - \langle \partial_x f_k'(x_k, a) \rangle f_{k-1}(x_{k-1}, a)$$

with $\langle x \rangle = \sum x_i / n$ and $\partial_x f_k(x_k, a) = \left[\frac{d}{dx} f_k(x_{k,i}, a_i) \right]_i$

Approximation of so-called message passing algorithms in statistical physics

Assuming

$$\mu^{a, x_0} \xrightarrow[n \to \infty]{\text{a.s.}} \mathscr{L}(\bar{a}, \bar{x}) \quad \text{in (say) } \mathcal{P}_2(\mathbb{R}^2),$$

$$\mu^{a, x_1, x_2, \ldots, x_k} \xrightarrow[n \to \infty]{\text{a.s.}} \mathscr{L}(\bar{a}) \otimes \mathcal{N}(0, R_k) \quad \text{in } \mathcal{P}_2(\mathbb{R}^{k+1})$$

where the sequence of covariance matrices (R_k) is constructed recursively according to the **State Evolution (SE) equations**

(Bayati, Montanari, Bolthausen, ... $\sim$ 2010)

SE equations

Let $(Z_1, Z_2, \ldots)$ with $\mathscr{L}((Z_1, \ldots, Z_k)) = \mathcal{N}(0, R_k)$ and $(Z_1, Z_2, \ldots) \perp\!\!\!\perp (\bar{a}, \bar{x})$.

Recursion:

$$R_1 = \mathbb{E} f_0(\bar{x}, \bar{a})^2$$

...

Knowing $R_k = \mathbb{C}\mathrm{ov}(Z_1, \ldots, Z_k)$, we have

$$\mathbb{E}\left[Z_{k+1} Z_\ell\right] = \begin{cases} \mathbb{E} f_k(Z_k, \bar{a}) f_{\ell-1}(Z_{\ell-1}, \bar{a}) & \text{if } \ell = 2, \ldots, k+1 \\ \mathbb{E} f_k(Z_k, \bar{a}) f_0(\bar{x}, \bar{a}) & \text{if } \ell = 1 \end{cases}$$

$\Rightarrow R_{k+1}$

$\mu^{u_\star}$ asymptotics: proof with AMP

Set $\alpha > 0$ and $a \in \mathbb{R}_+^n$ to be specified later. Put $f_0(x, a) = f_1(x, a) = \cdots = f(x, a) = \alpha(x + a)_+$.

AMP algorithm

$$x_{k+1} = \alpha G(x_k + a)_+ - \alpha^2 \gamma_k (x_{k-1} + a)_+ \quad \text{with}$$
$$\gamma_k = \langle \partial_x (x_k + a)_+ \rangle = \langle \mathbb{1}_{x_k + a > 0} \rangle$$

Key observation: Using an idea of [Donoho-Montanari'13], [Montanari-Richard'16], we can show that (Z_k, Z_{k+1}) in the SE equations become more and more correlated:

$$\mathbb{C}\mathrm{orr}(Z_k, Z_{k+1}) \xrightarrow[k \to \infty]{} 1,$$

which implies that

$$\lim_k \operatorname*{aslim}_n \frac{\langle x_{k+1}, x_k \rangle}{\|x_{k+1}\| \|x_k\|} = 1.$$

Proof with AMP (cont'd)

Thus,

$$x_k = \alpha G(x_k + a)_+ - \alpha^2 \gamma_k (x_k + a)_+ + \varepsilon_k$$

where $\lim_k \operatorname{aslim}_n \|\varepsilon_k\|^2/n = 0$. We rewrite this

$$x_k + a = \alpha G(x_k + a)_+ - \alpha^2 \gamma_k (x_k + a)_+ + a + \varepsilon_k, \text{ or}$$
$$-(x_k + a)_- = \alpha G(x_k + a)_+ - (1 + \alpha^2 \gamma_k)(x_k + a)_+ + a + \varepsilon_k, \text{ or}$$
$$-\frac{(x_k + a)_-}{1 + \alpha^2 \gamma_k} = \left(\frac{\alpha}{1 + \alpha^2 \gamma_k} G - I \right) (x_k + a)_+ + \frac{a}{1 + \alpha^2 \gamma_k} + \varepsilon'_k$$

in other words,

$$(x_k + a)_+ = \mathrm{LCP}\left(\frac{\alpha}{1 + \alpha^2 \gamma_k} G - I, \frac{a}{1 + \alpha^2 \gamma_k} + \varepsilon'_k \right).$$

Remember that

$$u_\star = \mathrm{LCP}(\beta G - I, r)$$

Identifying the two, using LCP perturbation results + SE equations $\Rightarrow$ the theorem

A more involved interaction matrix model

New interaction matrix model

Non-necessarily Gaussian, centered, variance profile, pairwise correlations, and sparse

$$Y_{ij} = \sqrt{v_{ij}} X_{ij}$$

- $\mathbb{E}X_{ij} = 0$, $\mathbb{E}X_{ij}^2 = 1$ and $\left(\mathbb{E}|X_{ij}|^k\right)^{1/k} \leq Ck^{\eta/2}$
- Elements of $\{X_{ii}, (X_{ij}, X_{ji})_{i<j}\}$ are independent
- $\mathbb{C}\mathrm{orr}(X_{ij}, X_{ji}) = \tau_{ij} \in [-1, 1]$ (correlation profile).

$V = [v_{ij}] \succcurlyeq 0$ is the variance profile matrix. For $K_n \gtrsim (\log n)^{\eta \vee 1}$,

- $\#$ non-zero elements per row $\leq \mathrm{Cst} \times K_n$
- $v_{ij} \leq \mathrm{Cst}/K_n$
- All row sums $\geq \mathrm{Cst}$

Ecological interpretation

- Couples of pairwise interactions are centered and independent
- Pairwise correlations specific to couples (i, j), reminiscent of the well-known elliptic model:
  $\tau_{ij} = 1$: often models competitive or mutualistic interactions
  $\tau_{ij} = -1$: predator-prey
  $\tau_{ij} = 0$: uncorrelated interactions
- Variance profile V: inhomogeneous interaction strengthes
- Sparsity: every species interacts with a small proportion, K_n/n, of other species

Asymptotic behavior of $\mu^{u_\star}$

No assumed structure on the variance profile or the correlation profile matrices $V = \left[v_{ij}\right]$ and $T = \left[\tau_{ij}\right]$

Therefore, $\mu^{u_\star}$ has **no reason to converge**

However, we can show that there exists a **deterministic sequence** $(\boldsymbol{\mu}_n)$ of probability measures that approximates $\mu^{u_\star}$ for large n

The parameters of a measure $\boldsymbol{\mu}_n$ will be obtained through the solution of a large **system of equations**

With additional assumptions, this system can be reduced to two integral equations

Result: system of equations

Additional assumptions:

- r is deterministic
- $\limsup_n \|(Y + Y^\mathsf{T})/2\| < 1$ a.s. Conditions for this in terms of matrices V and T can be deduced from literature, *e.g.*, [Bandeira-Van Handel'16]
- V and $Q = \left[\sqrt{v_{ij} v_{ji}} \tau_{ij}\right]_{i,j=1}^{n}$ have row sum norms $< 1/4$

System of equations

For each integer $n > 0$, let $Z \sim \mathcal{N}(0, I_n)$. The system of $2n$ equations in $(p, \zeta) \in \mathbb{R}_+^n \times [-1, 1]^n$

$$\begin{cases} p = V \operatorname{diag}(1+\zeta)^2 \, \mathbb{E}\left(\sqrt{p} \odot Z + r\right)_+^2 \\ \zeta = \operatorname{diag}(1+\zeta) Q \operatorname{diag}(1+\zeta) \, \mathbb{P}\left[\sqrt{p} \odot Z + r \geq 0\right] \end{cases}$$

admits an unique solution.

Result: $\mu^{u_\star} \sim$ mixture of truncated Gaussians

Theorem [GHN'24]

Let ξ be the Gaussian vector

$$\xi = \left[\xi_i\right]_{i\in[n]} = \mathrm{diag}(1+\zeta)\left(\sqrt{p}\odot Z + r\right)$$

and define the deterministic probability measure $\boldsymbol{\mu}_n = \mathscr{L}\left((\xi_\theta)_+\right)$, where θ is a uniformly distributed random variable on $[n]$, independent of Z. Then,

$$\mathrm{dist}_2\left(\mu^{u_\star}, \boldsymbol{\mu}_n\right) \xrightarrow[n\to\infty]{\mathcal{P}} 0$$

(distance in the space $\mathcal{P}_2(\mathbb{R})$)

Proof: a new AMP algorithm

New AMP for the generalized elliptic sparse model with a variance profile.
Might be helpful in contexts other than ecology, *e.g.*, in learning theory

Measurement matrix: $W = S^{\odot 1/2} \odot X$, where

- X is a above (with a correlation profile)
- $S = [s_{ij}] \succcurlyeq 0$ has same properties as V above:
  For $K_n \gtrsim (\log n)^{\eta \vee 1}$,
  - # non-zero elements per row $\leq \text{Cst} \times K_n$
  - $s_{ij} \leq \text{Cst}/K_n$
  - All row sums $\geq \text{Cst}$

Algorithm

$f_0, f_1, \ldots$ sequence of $\mathbb{R}^2 \to \mathbb{R}$ functions with sufficient regularity conditions

Deterministic parameter vector $a \in \mathbb{R}^n$ and initial value vector $x_0 \in \mathbb{R}^n$.

Algorithm:

$$x_{k+1} = \left[x_{k+1,i}\right] = Wf_k(x_k, a) - \operatorname{diag}\left(B\,\partial f_k(x_k, a)\right) f_{k-1}(x_{k-1}, a)$$

$B = \left[\sqrt{s_{ij}s_{ji}}\tau_{ij}\right]$ and $\partial f_k(x_k, a) = \left[\frac{d}{dx} f_k(x_{k,i}, a_i)\right]_{i \in [n]}$.

Result

We define a centered Gaussian family $(Z_{k,i})_{k \geq 1, i \in [n]}$ through appropriate DE equations (below)

Theorem

For each $k \geq 1$ and each continuous test function φ with quadratic growth at most,

$$\frac{1}{n} \sum_{i \in [n]} \varphi(a_i, x_{1,i}, \ldots, x_{k,i}) - \mathbb{E}\varphi(a_i, Z_{1,i}, \ldots, Z_{k,i}) \xrightarrow[n \to \infty]{\mathcal{P}} 0$$

(amounts to asymptotic behavior of $\mu^{a, x_1, \ldots, x_k}$ in probability in the $\mathcal{P}_2$ space)

A **combinatorial proof** along the lines of [Bayati-Lelarge-Montanari'15] building on the message passing structure of the algorithm.

Thank you

Questions?

The DE equations for our model

Distribution of the centered Gaussian family $(Z_{k,i})_{k\geq 1, i\in[n]}$.

The n sequences $Z_i = (Z_{k,i})_{k\geq 1}$ are independent. The covariance matrices R_i^k of the vectors $\vec{Z}_i^k = [Z_{1,i}, \ldots, Z_{k,i}]$ are defined recursively in k as follows.

$$R_i^1 = \sum_{\ell\in[n]} s_{i\ell} f_0(x_{0,i}, a_i)^2 \quad \text{for } i = 1, \ldots, n$$

$$\ldots$$

Given R_i^k for $i = 1, \ldots, n$,

$$H_i^k = \mathbb{E} \begin{bmatrix} f_0(x_{0,i}, a_i) \\ f_1(Z_{1,i}, a_i) \\ \vdots \\ f_k(Z_{k,i}, a_i) \end{bmatrix} \begin{bmatrix} f_0(x_{0,i}, a_i) & f_1(Z_{1,i}, a_i) & \cdots & f_k(Z_{k,i}, a_i) \end{bmatrix}$$

$$R_i^{k+1} = \sum_{\ell=1}^{n} s_{i\ell} H_\ell^k \quad \text{for } i = 1, \ldots, n$$